Noise Risk and Derivative Price*

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Abstract

This paper considers a static asset market with dependent background risk, which is described as regression dependence. We examine a condition of preferences to determine if dependent background risk decreases equilibrium asset prices. In such a condition, absolute risk aversion decreases and relative risk aversion is less than unity.

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1 Introduction

Many studies are devoted to determine conditions on preference and risk changes for optimal portfolios to monotonically change. It is very natural that this research extends to asset prices in pure exchange economies since optimal portfolios lead them. Gollier and Schlesinger (2002) introduced a comparative static technique for asset prices based on their excess demand functions and showed that many comparative static results of optimal portfolios can extend to asset prices. Gollier and Schlesinger (2002) considered an economy where a single risky asset is traded, which can be viewed as the market portfolio in the economy. However, there are many types of derivatives written on the market portfolio return in actual asset markets. This suggests that we also need to investigate how changes in the riskiness of the market portfolio return influences derivative prices.

This note determines a sufficient condition on preference, return and wealth function shapes to guarantee that noise risk monotonically changes derivative prices. Our analysis is motivated by Gollier and Schlesinger (2002) and extends their work on asset markets without derivatives to them with derivatives. Gollier and Schlesinger (1996) determined a condition on preference to guarantee that noise risk decreases optimal portfolios, and Gollier and Schlesinger (2002) extended them to asset prices. This is an interesting analysis for the following two reasons. First, noise risk is a special case of the notion of the increase in risk introduced by Rothschild and Stiglitz (1970). Second, it provides an alternative explanation for the miscalibration problem of asset prices observed by Mehra and Prescott (1985). Their analysis owes much to the linearity. The introduction of derivatives written on the market portfolio return implies that both derivative return and wealth are nonlinear functions. So it is not straightforward to expand their analysis to asset markets with derivatives.

The organization of our note is as follows. In Section 2, we give equilibrium derivative prices. Section 3 provides our main result and its remarks. The proof is in Appendix. Section 4 summarizes our analysis and describes another possible application.

2 Asset Market

Let us consider a static version of Lucas (1978) economy, that is, a two–date pure exchange economy with a representative investor. The investor has a (von Neumann–Morgenstern) utility function $u : \mathbb{R} \to \mathbb{R}$, which is strictly increasing and concave. It is assumed that the utility function is sufficiently smooth for the simplicity of the analysis. “Sufficiently” means that all of the required higher order derivatives in the analysis exist. There are one risk–free asset and N risky assets in the asset market.

The risk–free asset is the numeraire, and its (gross) risk–free rate is normalized to one without loss of any generality. One risky asset is the market portfolio, and its return is a random variable $r_1(\tilde{x}) = \tilde{x}$ with Cumulartive Distribution Function (CDF) $F : [\underline{x}, \overline{x}] \to [0, 1]$. We assume that CDF F is differentiable, that is, the associated Probability Density Function (PDF) $f(x) = F'(x)$ exists, where the prime denotes derivative. Other risky assets are derivatives written on the market portfolio return. Then their returns are functions of the market portfolio return, $r_n(\tilde{x}),\ n = 2, 3, \ldots, N$. The investor is endowed with w units of the risk–free asset and one unit of each risky asset, respectively. We note that the market portfolio can be viewed as a derivative written on itself return.

The investor buys the portfolio $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_N)$ to maximize the expected utility from the final wealth, α_0 is the quantity of the risk–free asset and α_n is the quantity of derivative n. The risk–free asset price is equal to one because of its normalization, and the price of derivative n is denoted as q_n. The expected utility maximization problem for the investor is given as follows:

$$\begin{aligned} \mathbf{P} \quad V(\boldsymbol{\alpha}) &:= \max_{\{\boldsymbol{\alpha}\}} \ \mathbb{E}[u(\alpha_0 + \sum_{n=1}^{N} \alpha_n r_n(\tilde{x}))] \\ \text{s.t. } \alpha_0 &+ \sum_{n=1}^{N} \alpha_n q_n \le w + \sum_{n=1}^{N} q_n \end{aligned} \tag{1}$$

Let us define the Lagrangean $\mathcal{L}(\boldsymbol{\alpha}, \lambda) := \mathbb{E}[u(\alpha_0 + \sum_{n=1}^{N} \alpha_n r_n(\tilde{x}))] + \lambda(w + \sum_{n=1}^{N} \alpha_n q_n - \alpha_0 - \sum_{n=1}^{N} \alpha_n q_n)$. The existence of the representative investor means that the no–trade equilibrium occurs in the asset market, and so the market clearing condition is given by $\alpha_0 = w$ and $\alpha_n = 1$. The first–order necessary conditions of the maximization problem are as follows:

$$\frac{\partial \mathcal{L}}{\partial \alpha_0} = \mathbb{E}[u'(z(\tilde{x}))] - \lambda = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \alpha_n} = \mathbb{E}[r_n(\tilde{x}) u'(z(\tilde{x}))] - \lambda p_n = 0,\ n = 1, 2, \ldots, N, \tag{3}$$

where $z(x) = w + \sum_{n=1}^{N} r_n(x)$ is the final wealth in equilibrium. Since the objective function is concave and the constraint is linear, the first–order necessary conditions also meet the second–order sufficient conditions. By equations (2) and (3), the equilibrium price of derivative n is given as follows:

$$q_n = \frac{\mathbb{E}[r_n(\tilde{x}) u'(z(\tilde{x}))]}{\mathbb{E}[u'(z(\tilde{x}))]}. \tag{4}$$

Next, we consider the case in which noise risk is added to the market portfolio return. Noise risk is a zero–mean random variable $\tilde{\epsilon}$ with a differentiable CDF

$G : [\underline{\epsilon}, \overline{\epsilon}] \to [0,1]$. We also assume that the noise risk is statistically independent from the market portfolio return, that is $\tilde{x}$ and $\tilde{\epsilon}$ are mutually independent random variables. Since the market portfolio return is given as $\tilde{x} + \tilde{\epsilon}$, the equilibrium price of derivative n is given by a similar analysis as follows:

$$Q_n = \frac{\mathbb{E}[r_n(\tilde{x} + \tilde{\epsilon})u'(z(\tilde{x} + \tilde{\epsilon}))]}{\mathbb{E}[u'(z(\tilde{x} + \tilde{\epsilon}))]} \tag{5}$$

Here, $z(x + \epsilon) = w + \sum_{n=1}^{N} r_n(x + \epsilon)$ is the final wealth in equilibrium. In the reminder of this note, we determine a condition on preference, return and wealth function shapes to guarantee that noise risk monotonically changes derivative prices. We suppress subscript n in the reminder of this note for notational ease.

3 Main Result

3.1 Theorem

Before giving the theorem, we define (Arrow–Pratt absolute) risk aversion as $\mathcal{A}(x) = -u''(x)/u'(x)$. The proof is in Appendix.

Theorem 3.1.
Suppose that risk aversion is decreasing and convex, $\mathcal{A}'(x) \leq 0$ and $\mathcal{A}''(x) \geq 0$. Also suppose that the first–order derivative of wealth function is positive, decreasing and concave, $z'(x) \geq 0$, $z''(x) \leq 0$ and $z'''(x) \leq 0$. Noise risk increases (decreases) prices of derivatives whose returns are decreasing and convex (increasing and concave), $r'(x) \leq (\geq)\ 0$ and $r''(x) \geq (\leq)\ 0$.

3.2 Remark

We provide two remarks on this theorem. The first remark is on the condition of utility function and the second remark is on that of the wealth function shape.

3.2.1 The First Remark

Gollier and Pratt (1996) determined a condition to guarantee that any unfair background risk makes decision-makers in a more risk-averse manner. They named it risk vulnerability. Gollier and Schlesinger (2002) applied this to determine the effect of noise risk on asset price because of the linerity of the wealth function. We cannot apply it to our analysis because of the nonlinearity of the wealth function. Gollier and Pratt (1996) also gave sufficient conditions for risk vulnerability which have natural economic interpretations. One of them is decreasing and convex risk aversion which appeared in the above theorem. This is an interesting and possibly surprising thing, since our analysis is quite different from Gollier and Schlesinger (2002) because of the nonlinearity.

3.2.2 The Second Remark

Let us consider derivatives written on the market portfolio return as vanilla options, *i.e.*, they are a call option whose return is $\max\{x - K, 0\}$ or a put option whose return is $\max\{K - x, 0\}$, where K is a strike price. Since the sum of piece–wise linear functions is also piece–wise linear, the wealth function is piece–wise linear. This means $z'''(x) = 0$. We note that $z'''(x) = 0$ is included in $z'''(x) \leq 0$. Most derivative returns traded in actual markets are piece–wise linear functions of their underlying asset returns. Then, the condition $z'''(x) \leq 0$ is not too restrictive from the practical viewpoint. Next, we discuss the concavity of the wealth function. Since both call and put option returns are convex functions and the sum of convex functions is also convex, the wealth function is convex when supply of all call and put options is positive. This is inconsistent with the concavity of the wealth function. Since a negative convex function is concave, a negative supply of derivatives leads to the concavity of their return. Then we need to assume negative supply of some call and put options to guarantee the concavity of the wealth function. In other words, we implicitly assume that there are risk–takers outside the economy who supply some call and put options.

Last, we mention the comparative static predictions obtained by theorem 3.1. Since linear functions are concave, we obtain the following result:

Result
Assume that the conditions of theorem 3.1 are satisfied. Noise risk increases the market portfolio price.

Gollier and Schlesinger (2002) introduced a comparative statics technique for asset prices based on their excess demand functions, and obtained various comparative statics results. One of their results is that noise risk increases market portfolio prices.[1)] This is an alternative explanation of the equity premium puzzle observed Mehra and Prescott (1985). In the economy with vanilla options, decreasing and convex risk aversion have the comparative static prediction to resolve the equity premium puzzle.

Since put option returns are decreasing and convex, we have the following comparative statics prediction:

[1)] They consider economies satisfying the one–fund separation theorem. We note that increase in the risky asset price is identical with decrease in the equity premium in the economy of which numeraire is the risk–free asset.

Result
Assume that the conditions of theorem 3.1 are satisfied. Noise risk decreases put option prema if their supply is positive, and increases that if their supply is negative.

4 Conclusion

This note gives a sufficient condition to guarantee that noise risk monotonically changes derivative prices. That condition is decreasing and convex risk aversion. As in Gollier and Pratt (1996), this is a sufficient condition of risk vulnerability and has natural economic interpretations. This is an interesting and possibly surprising result because we cannot apply the result of Gollier and Pratt (1996) to our analysis. The conditions on return and wealth functions may be too restrictive. This suggests a direction of our future research.

We consider an asset market consisting of the risk–free asset, the market portfolio and derivatives written on the market portfolio return. Our analysis can be applied to the hedging decision problem of competitive firms originated by Sandmo (1971) with slight modifications. This application is interesting because put options are hedging instrument that are widely analyzed and used.

Appendix

A Proof

Since the logic of the proof is same except sign changes, we give the proof for the first statement, that is, the case of derivatives with decreasing and convex returns.

Our proof consists of the following two steps:

$$\textbf{1}.\ \ Q = \frac{\mathbb{E}[r(\tilde{x}+\tilde{\epsilon})u'(z(\tilde{x}+\tilde{\epsilon}))]}{\mathbb{E}[u'(z(\tilde{x}+\tilde{\epsilon}))]} \geq \frac{\mathbb{E}[r(\tilde{x})u'(z(\tilde{x}+\tilde{\epsilon}))]}{\mathbb{E}[u'(z(\tilde{x}+\tilde{\epsilon}))]} =: p, \tag{6}$$

$$\textbf{2}.\ \ p = \frac{\mathbb{E}[r(\tilde{x})u'(z(\tilde{x}+\tilde{\epsilon}))]}{\mathbb{E}[u'(z(\tilde{x}+\tilde{\epsilon}))]} \geq \frac{\mathbb{E}[r(\tilde{x})u'(z(\tilde{x}))]}{\mathbb{E}[u'(z(\tilde{x}))]} = q. \tag{7}$$

Combining the above inequalities, we obtain the inequality displaying the statement of the theorem,

$$Q = \frac{\mathbb{E}[r(\tilde{x}+\tilde{\epsilon})u'(z(\tilde{x}+\tilde{\epsilon}))]}{\mathbb{E}[u'(z(\tilde{x}+\tilde{\epsilon}))]} \geq \frac{\mathbb{E}[r(\tilde{x})u'(z(\tilde{x}))]}{\mathbb{E}[u'(z(\tilde{x}))]} = q. \tag{8}$$

The first step

We give the statement of the first step as follows:

Claim A.1.

$$Q = \frac{\mathbb{E}[r(\tilde{x}+\tilde{\epsilon})u'(z(\tilde{x}+\tilde{\epsilon}))]}{E[u'(z(\tilde{x}+\tilde{\epsilon}))]} \geq \frac{\mathbb{E}[r(\tilde{x})u'(z(\tilde{x}+\tilde{\epsilon}))]}{E[u'(z(\tilde{x}+\tilde{\epsilon}))]} = p, \tag{9}$$

for all derivatives with decreasing and convex returns.

Proof. Let us define the function

$$\hat{h}(x,\epsilon) := \frac{u'(z(x+\epsilon))f(x)g(\epsilon)}{\mathbb{E}[u'(z(\tilde{x}+\tilde{\epsilon}))]}. \tag{10}$$

Since $\int_{\underline{\epsilon}}^{\overline{\epsilon}} \hat{h}(x,\epsilon)dxd\epsilon = 1$ and $\hat{h}(x,\epsilon) > 0,\ \forall(x,\epsilon) \in [\underline{x},\overline{x}] \times [\underline{\epsilon},\overline{\epsilon}]$, $\hat{h}(x,\epsilon)$ is a joint PDF of x and ϵ defined over support $[\underline{x},\overline{x}] \times [\underline{\epsilon},\overline{\epsilon}]$. Using PDF $\hat{h}$, we have that

$$\begin{aligned} Q &= \frac{\mathbb{E}[r(\tilde{x}+\tilde{\epsilon})u'(z(\tilde{x}+\tilde{\epsilon}))]}{\mathbb{E}[u'(z(\tilde{x}+\tilde{\epsilon}))]} \\ &= \int_{\underline{\epsilon}}^{\overline{\epsilon}}\int_{\underline{x}}^{\overline{x}} r(x+\epsilon)\frac{u'(z(x+\epsilon))f(x)g(\epsilon)}{\mathbb{E}[u'(z(\tilde{x}+\tilde{\epsilon}))]}dxd\epsilon \\ &= \int_{\underline{\epsilon}}^{\overline{\epsilon}}\int_{\underline{x}}^{\overline{x}} r(x+\epsilon)\hat{h}(x,\epsilon)dxd\epsilon \\ &= \hat{\mathbb{E}}[r(\tilde{x}+\tilde{\epsilon})], \end{aligned} \tag{11}$$

where $\hat{\mathbb{E}}$ denotes the expectation operator with respect to PDF $\hat{h}$. In a similar way, we have that

$$p = \frac{\mathbb{E}[r(\tilde{x})u'(z(\tilde{x}+\tilde{\epsilon}))]}{\mathbb{E}[u'(z(\tilde{x}+\tilde{\epsilon}))]} = \hat{\mathbb{E}}[r(\tilde{x})]. \tag{12}$$

We show that $\hat{\mathbb{E}}[r(x+\tilde{\epsilon}) \mid x] \geq r(x)$ for all $x \in [\underline{x}, \overline{x}]$ to obtain the claim. By the Jensen's inequality and r is a convex function, we have that

$$\hat{\mathbb{E}}[r(x+\tilde{\epsilon}) \mid x] \geq r(x + \hat{\mathbb{E}}[\tilde{\epsilon} \mid x]). \tag{13}$$

Conditional PDF $\hat{h}(\epsilon \mid x)$ is given as

$$\begin{aligned}\hat{h}(\epsilon \mid x) = \frac{\hat{h}(x,\epsilon)}{\hat{h}(x)} &= \frac{u'(z(x+\tilde{\epsilon}))f(x)g(\epsilon)}{\mathbb{E}[u'(z(\tilde{x}+\tilde{\epsilon}))]} \frac{\mathbb{E}[u'(z(\tilde{x}+\tilde{\epsilon}))]}{\mathbb{E}[u'(z(x+\tilde{\epsilon}))]f(x)} \\ &= \frac{u'(z(x+\tilde{\epsilon}))g(\epsilon)}{\mathbb{E}[u'(z(x+\tilde{\epsilon}))]}\end{aligned} \tag{14}$$

We have that

$$\begin{aligned}\frac{h(\epsilon_t)}{h(\epsilon_s)} &= \frac{u'(z(x+\epsilon_t))g(\epsilon_t)}{\mathbb{E}[u'(z(x+\tilde{\epsilon}))]} \frac{u'(z(x+\epsilon_s))g(\epsilon_s)}{\mathbb{E}[u'(z(x+\tilde{\epsilon}))]} \\ &= \frac{u'(z(x+\epsilon_t))}{u'(z(x+\epsilon_s))} \frac{g(\epsilon_t)}{g(\epsilon_s)} \leq \frac{g(\epsilon_t)}{g(\epsilon_s)}\end{aligned} \tag{15}$$

for all $\epsilon_s, \epsilon_t \in [\underline{\epsilon}, \overline{\epsilon}]$ with $\epsilon_s \leq \epsilon_t$. The inequality comes from $u'(z(x+\epsilon_s)) \geq u'(z(x+\epsilon_t))$, since $u' \circ z$ is a decreasing function. The above inequality is equivalent to CDF $\hat{H}(\epsilon \mid x) = \int_{\underline{\epsilon}}^{\epsilon} \hat{h}(\epsilon \mid x) d\epsilon$ is dominated by $G(\epsilon)$ in the sense of Monotone Likelihood Ratio Dominance (MLRD). Since MLRD means First–order Stochastic Dominance (FSD), this implies $0 = \mathbb{E}[\tilde{\epsilon}] \geq \hat{\mathbb{E}}[\tilde{\epsilon}]$, and we have that

$$r(x + \hat{\mathbb{E}}[\tilde{\epsilon} \mid x]) \geq r(x). \tag{16}$$

Combining the inequalities (13) and (16), we have that $\mathbb{E}[r(x+\tilde{\epsilon}) \mid x] \geq r(x)$ for all $x \in [\underline{x}, \overline{x}]$. We obtain that

$$Q = \int_{\underline{x}}^{\overline{x}} \hat{\mathbb{E}}[r(x+\tilde{\epsilon}) \mid x]\hat{h}(x)dx \geq \int_{\underline{x}}^{\overline{x}} r(x)\hat{h}(x)dx = p. \tag{17}$$

□

The second step

We give the statement of the second step as follows:

Claim A.2. Suppose that risk aversion is decreasing and convex, $\mathcal{A}'(x) \leq 0$ and $\mathcal{A}'' \geq 0$. Also suppose that the first–order derivative of the wealth function is positive, decreasing and concave, $z'(x) \geq 0$, $z''(x) \leq 0$ and $z'''(x) \leq 0$.

$$p = \frac{\mathbb{E}[r(\tilde{x})u'(z(\tilde{x}+\tilde{\epsilon}))]}{E[u'(z(\tilde{x}+\tilde{\epsilon}))]} \geq \frac{\mathbb{E}[r(\tilde{x})u'(z(\tilde{x}))]}{E[u'(z(\tilde{x})]} = q, \tag{18}$$

for all derivatives with decreasing and convex returns.

Proof. Let us define the function

$$f^*(x,\epsilon) := \frac{\mathbb{E}[u'(z(x+\epsilon))]f(x)}{\mathbb{E}[u'(z(\tilde{x}+\tilde{\epsilon}))]}, \tag{19}$$

and

$$f^{**}(x,\epsilon) := \frac{u'(z(x))f(x)}{\mathbb{E}[u'(z(\tilde{x})]}, \tag{20}$$

Since $\int_{\underline{x}}^{\overline{x}} f^*(x)dx = 1$ and $f^*(x) > 0$, $\forall(x) \in [\underline{x}, \overline{x}]$, $f^*(x)$ is a PDF defined over support $[\underline{x}, \overline{x}]$. $f^{**}(x)$ is also a PDF defined over support $[\underline{x}, \overline{x}]$. Using PDF $\hat{f}^*$, we have that

$$\begin{aligned} p &= \frac{\mathbb{E}[r(\tilde{x})u'(z(\tilde{x}+\tilde{\epsilon}))]}{\mathbb{E}[u'(z(\tilde{x}+\tilde{\epsilon}))]} \\ &= \int_{\underline{x}}^{\overline{x}} r(x)\frac{\mathbb{E}[u'(z(x+\tilde{\epsilon}))]f(x)}{\mathbb{E}[u'(z(\tilde{x}+\tilde{\epsilon}))]}dx \\ &= \int_{\underline{x}}^{\overline{x}} r(x)f^*(x)dx \\ &= \mathbb{E}^*[r(\tilde{x})], \end{aligned} \tag{21}$$

where $\mathbb{E}^*$ denotes the expectation operator with respect to PDF f^*. In a similar way, we have

$$q = \frac{\mathbb{E}[r(\tilde{x})u'(z(\tilde{x})]}{\mathbb{E}[u'(z(\tilde{x})]} = \mathbb{E}^{**}[r(\tilde{x})]. \tag{22}$$

We prove that CDF F^{**} dominates F^* in the sense of MLRD. Since MLRD implies FSD, we obtain the claim, $p = \mathbb{E}^*[r(\tilde{x})] \geq \mathbb{E}^{**}[r(\tilde{x})] = q$ because of r being decreasing in x.

F^{**} dominates F^* in the sense of MLRD, if $f^{**}(y)/f^{**}(x) \geq f^*(y)/f^*(x)$ for all $x, y \in [\underline{x}, \overline{x}]$ with $x \leq y$. We have that

$$\frac{f^{**}(y)}{f^*(y)} \geq \frac{f^{**}(x)}{f^*(x)} \Leftrightarrow \frac{u'(z(y))}{\mathbb{E}[u'(z(y+\tilde{\epsilon}))]} \geq \frac{u'(z(x))}{\mathbb{E}[u'(z(x+\tilde{\epsilon}))]}. \tag{23}$$

This is equivalent to

$$\begin{aligned} &\frac{\partial}{\partial x}\left(\frac{u'(z(x))}{\mathbb{E}[u'(z(x+\tilde{\epsilon})))]}\right) \geq 0 \Leftrightarrow \\ &u''(z(x))z'(x)\mathbb{E}[u'(z(x+\tilde{\epsilon}))] > u'(z(x))\mathbb{E}[u'(z(x+\tilde{\epsilon}))z'(x+\tilde{\epsilon})] \end{aligned} \tag{24}$$

for all $x \in [\underline{x}, \overline{x}]$. Since u'' is increasing and z' is decreasing[2)],

$$\mathbb{E}[u'(z(x+\tilde{\epsilon}))]\mathbb{E}[z'(x+\tilde{\epsilon})] \geq \mathbb{E}[u'(z(x+\tilde{\epsilon}))z'(x+\tilde{\epsilon})]. \tag{25}$$

This implies that

$$u'(z(x))\mathbb{E}[u'(z(x+\tilde{\epsilon}))]\mathbb{E}[z'(x+\tilde{\epsilon})] \geq u'(z(x))\mathbb{E}[u'(z(x+\tilde{\epsilon}))z'(x+\tilde{\epsilon})]. \tag{26}$$

Then, if we have the following inequality,

$$\begin{aligned} u''(z(x))z'(x)\mathbb{E}[u'(z(x+\tilde{\epsilon}))] &\geq u'(z(x))\mathbb{E}[u'(z(x+\tilde{\epsilon}))]\mathbb{E}[z'(x+\tilde{\epsilon})] \\ \Leftrightarrow \frac{u''(z(x))}{u'(z(x))} &\geq \frac{\mathbb{E}[u''(z(x+\tilde{\epsilon}))]}{\mathbb{E}[u'(z(x+\tilde{\epsilon}))]}\frac{\mathbb{E}[z'(x+\tilde{\epsilon})]}{z'(x)}, \end{aligned} \tag{27}$$

hence we also have the inequaltiy (24).

By Jensen's inequality, $z''' \leq 0$ means that

$$\mathbb{E}[z'(x+\tilde{\epsilon})] \leq z'(x+\mathbb{E}[\tilde{\epsilon}]) = z'(x) \Leftrightarrow \frac{\mathbb{E}[z'(x+\tilde{\epsilon})]}{z'(x)} \leq 1. \tag{28}$$

This means that

$$\frac{u''(z(x))}{u'(z(x))} \geq \frac{\mathbb{E}[u''(z(x+\tilde{\epsilon}))]}{\mathbb{E}[u'(z(x+\tilde{\epsilon}))]} \tag{29}$$

is sufficient for the inequality (27). Using PDF $\hat{g}(\epsilon) = u'(z(x+\epsilon))g(\epsilon)/\mathbb{E}[u'(z(x+\tilde{\epsilon}))]$, we have that

$$\begin{aligned} \frac{\mathbb{E}[u''(z(x+\tilde{\epsilon}))]}{\mathbb{E}[u'(z(x+\tilde{\epsilon}))]} &= \int_{\underline{\epsilon}}^{\overline{\epsilon}} -\frac{u''(z(x+\epsilon))}{u'(z(x+\epsilon))}\frac{u'(z(x+\epsilon))}{\mathbb{E}[u'(z(x+\tilde{\epsilon}))]} \\ &= \hat{\mathbb{E}}[\mathcal{A}(z(x+\tilde{\epsilon}))]. \end{aligned} \tag{30}$$

Since $\mathcal{A}$ is decreasing and convex, and z is concave in x, $\mathcal{A} \circ z$ is convex in x. Then, we have that

$$\hat{\mathbb{E}}[\mathcal{A}(z(x+\tilde{\epsilon}))] \geq \mathcal{A}(z(x+\hat{\mathbb{E}}[\epsilon])) \geq \mathcal{A}(z(x)). \tag{31}$$

The first inequality comes from the Jensen's inequality and the second inequality comes from $0 = \mathbb{E}[\tilde{\epsilon}] \geq \hat{\mathbb{E}}[\tilde{\epsilon}]$ by a similar discussion appeared in the proof of the first step. Since we have the inequality (29), CDF F^{**} dominates F^{*} in the sense of MLRD. We complete the proof. □

[2)]Decreasing absolute risk aversion implies prudence, $u''' \geq 0$.

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